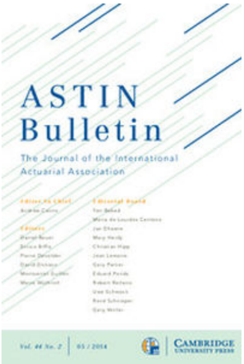


THE IMPACT OF INFLATION RISK ON FINANCIAL PLANNING AND RISK-RETURN PROFILES

Stefan Graf, Lena Haertel, Alexander Kling and Jochen Ruß

ASTIN Bulletin / Volume 44 / Issue 02 / May 2014, pp 335 - 365
DOI: 10.1017/asb.2014.1, Published online: 04 February 2014

THE IMPACT OF INFLATION RISK ON FINANCIAL PLANNING AND RISK-RETURN PROFILES

BY

STEFAN GRAF, LENA HAERTEL, ALEXANDER KLING AND JOCHEN RUß

ABSTRACT

The importance of funded private or occupational old-age provision is expected to increase due to demographic changes and the resulting problems for government-run pay-as-you-go systems. Clients and advisors therefore need reliable methodologies to match offered products with clients' needs and risk appetite. In Graf *et al.* (2012), the authors have introduced a methodology based on stochastic modeling to properly assess the risk-return profiles — i.e. the probability distribution of future benefits — of various old-age provision products. In this paper, we additionally consider the impact of inflation on the risk-return profile of old-age provision products. In a model with stochastic interest rates, stochastic inflation and equity returns including stochastic equity volatility, we derive risk-return-profiles for various types of existing unit-linked products with and without embedded guarantees and especially focus on the difference between nominal and real returns. We find that typical "rule of thumb" approximations for considering inflation risk are inappropriate and further show that products that are considered particularly safe by practitioners because of nominal guarantees may bear significant inflation risk. Finally, we propose product designs suitable to reduce inflation risk and investigate their risk-return profile in real terms.

KEYWORDS

Stochastic modeling, financial planning, inflation, product design.

1. INTRODUCTION

The demographic transition constitutes a severe challenge for government-run pay-as-you-go pension systems in many countries. Therefore, the importance of funded private and/or occupational old-age provision has been increasing and will very likely continue to increase. Competing for clients' money, product providers (e.g. life insurers and asset managers) have come up with a variety of

Astin Bulletin 44(2), 335-365. doi: 10.1017/asb.2014.1

"packaged products" for old-age provision that often consist of equity investments combined with certain nominal maturity value and/or minimum death benefit guarantees. Graf *et al.* (2012) provide an overview over the most important types of such products and also show that information that is typically provided by product providers is not sufficient to assess and to compare the downside risk and the return potential (the "risk-return profile") of different products. They also argue that results provided in the literature on financial planning (cf. the literature overview given in their paper) can often not be applied in practice. Therefore, they introduce a methodology suitable to be implemented in practice to assess the risk-return profile of such products. In a model with stochastic interest rates, equity returns and equity volatility, they derive risk-return-profiles — i.e. probability distributions of investment returns — for various products and investigate the impact of premium payment mode and investment horizon.

In the meantime, e.g. in Germany, several insurers, distribution organizations as well as product rating companies are using analyses based on risk-return profiles. Furthermore, a law has been passed in Germany that requires that in the future, for certain products the client needs to be informed about the "risk-return class" of a product, which has to be calculated based on risk-return profiles derived in a stochastic simulation framework.[1]

Graf *et al.* (2012) (and to the best of our knowledge all parties currently using risk-return profiles) are solely concerned with analyzing nominal returns. However, the relevant criterion should not be the number of units of some currency that is provided as a benefit but rather the purchasing power of the benefit. Hence, an appropriate assessment of inflation risk should be included.

This is currently of particular interest since recent quantitative easings issued by several governments in order to stimulate capital markets to overcome the recent subprime and current Euro debt crisis have revived the issue of a potential inflation and its impact on financial planning. For assessing the protection against inflation risk of different asset classes, the empirical analyses of Amenc *et al.* (2009) provide a good starting point. Further, recent academic literature deals with portfolio optimization problems explicitly taking inflation risk into account: Briére and Signori (2012) solve a portfolio allocation problem in a risk-reward setting focusing on real (i.e. inflation adjusted) investment return whereas Weiyin (2013) solves the asset allocation problem applying the expected utility approach and taking inflation into account.

However, these important theoretical results are often too complex for practical use by clients and/or advisors and highly depend on the particular choice of a utility function. Furthermore, the results are not applicable in practice because assumptions about available products are often oversimplifying. For example, charges are usually not considered at all and only single premium products are considered. Additionally, a practical implementation of such approaches would often require a continuous management of clients' accounts, which is often complex, not feasible for rather small contract volumes, or might in some countries result in tax disadvantages upon each transaction. Therefore, in practice, often "packaged products" where certain strategies are implemented that do not require any action on the client's side during the lifetime of the product, are offered and many financial advisors try to find the most suitable product out of a variety of such products for each client.

In the present paper, we include inflation risk in the framework suitable for practical implementation that has been proposed by Graf *et al.* (2012), and particularly analyze the difference between nominal and real returns observed. We analyze common old-age provision products particularly taking into account charges or regular premiums that occur in a practical application. Further, we especially focus on products that are considered particularly safe by practitioners because of certain investment guarantees on a nominal basis and find that they may bear a significant inflation risk. Therefore, we propose adjustments to the existing products based on asset allocations suitable to reduce inflation risk. A key finding of our paper is that inflation risk can be reduced significantly or even eliminated by suitable product designs.

In the academic literature, three approaches for a stochastic modeling of inflation are common. The probably most common approach usually applied in pricing is given by the "Jarrow–Yildirim" approach. Among others,[2] Jarrow and Yildirim (2003) propose a model based on the idea of linking nominal and real units of currency with a foreign exchange approach. They derive the dynamics of some consumer price index (CPI) along with the instantaneous nominal and real interest rates in a Heath-Jarrow-Morton framework. The CPI is then interpreted as the exchange rate between the nominal (i.e. domestic) and real (i.e. foreign) "currencies". Additionally, e.g. Belgrade *et al.* (2004) and Mercurio (2005) introduce alternative approaches, proposing the use of market models based on traded inflation derivatives similar to market models for interest rate modeling. Note that both, the Jarrow–Yildirim and the market model approaches, are primarily designed for pricing inflation-linked derivatives (under some risk-neutral probability measure) and may therefore not directly be applied to a (real-world) analysis under an objective probability measure. Finally, e.g. Ahlgrim *et al.* (2005) develop an economic scenario generator capable of simulating a variety of economic variables including inflation rates by using a one-factor diffusion model designed for analyses in the actuarial sector. Since we are also interested in real-world analyses, we adopt the approach introduced by Ahlgrim *et al.* (2005) in what follows.

The remainder of this paper is organized as follows. Section 2 introduces the products that are analyzed in a first step. We particularly investigate their nominal and real returns assuming the financial model as introduced in Section 3. Section 4 then provides a quantitative assessment of the resulting nominal and in particular real risk-return profiles. Since we find that the products introduced in Section 2 (including products with nominal investment guarantee) bear significant exposure to inflation risk, we propose product modifications that might be suitable to reduce inflation risk in Section 5. Section 6 carries out some sensitivity analyses to the obtained results and Section 7 finally concludes.

2. Standard Products

In the first part of our analyses, we consider several product types that are (sometimes in different variants) common in retirement planning and offered by various financial institutions such as banks, insurers or asset managers in many countries.[3] At the client's retirement date, the products either pay a lump sum benefit or — in the case of annuitization — are typically converted into a traditional life-long annuity calculated with some annuity conversion rate. Since the products are typically very similar in the payout phase, we focus on the accumulation phase, where significantly different products exist.

We present and model several products with and without nominal investment guarantees. As products without nominal guarantees, we look at a pure investment in an equity fund or in a fixed income instrument modeled as a zero-coupon bond. Also, we consider two main categories of products with nominal guarantees: products with a static hedge (underlying plus put) and products with a dynamic hedge (constant proportion portfolio insurance; CPPI). Variants of these products can be found in many countries. For example, so-called GMAB (Guaranteed Minimum Accumulation Benefit) guarantees in U.S.-style variable annuities[4] are a special case of the static-hedge product introduced in Section 2.2.1; so-called dynamic hybrid insurance products,[5] which are currently the most successful guaranteed unit-linked products in Germany are a variant of the dynamic hedge product introduced in Section 2.2.3. Finally, although rather trivial, the special (static) case of this product for a CPPI-multiplier of 1 is still frequently offered in many continental European countries and therefore considered separately in Section 2.2.2.

We consider products with single premium P and with regular monthly premium payment P and a term of T years. The following fee structure is applied to all products:

- Premium proportional charges $\beta \cdot P$ reduce the amount invested to $(1-\beta) \cdot P$.
- Account proportional charges, γ — quoted as an annual fee — are deducted on a monthly basis from the client's account.
- Additionally, fund management charges, c (also quoted as an annual charge but deducted daily), are applied within mutual funds if such funds are used in the packaged product. Additional guarantee charges may apply for the products with investment guarantees (see below).

With A_t denoting the client's account value at time t and $Perf_{t,t+\frac{1}{12}}$ denoting the performance of the considered products from t to $t+\frac{1}{12}$, we obtain $A_0 = (1-\beta) \cdot P$ and define the client's account value (immediately before the beginning of the next month) as

$$A\left(t + \frac{1}{12}\right) - = A_t \cdot Perf_{t,t+\frac{1}{12}} \cdot (1-\gamma)^{\frac{1}{12}}.$$

Further, the account value at the beginning of the month is then given as $A_{(t+\frac{1}{12})} = A_{(t+\frac{1}{12})-} + (1-\beta) \cdot P$ in the case of regular contributions and $A_{(t+\frac{1}{12})} = A_{(t+\frac{1}{12})-}$ in the case of a single contribution.

2.1. Mutual fund and fixed income investment

In these simple products, the client's account value A_t is either completely invested in an equity fund or in a fixed income instrument modeled as a zero-coupon bond with maturity T where $p(t, T)$ denotes the zero-bond's price at time t.

2.2. Products with nominal investment guarantee

The following products equipped with a "money back guarantee" provide the guarantee that at least the client's contributions are paid back at maturity.[6] However, the way of generating this guarantee varies throughout the considered products: we let G_t denote the so-called guarantee basis at time t which is however only valid at the contract's maturity T. We let $G_t = P \, \forall \, t$ for the single and

$$G_t = \begin{cases} ([t \cdot 12] + 1) \cdot P, & t < T \\ T \cdot 12 \cdot P, & t = T \end{cases}$$

for the regular contribution case.

2.2.1. *Static option-based product.* For this product, the premium is invested into an underlying fund (in our numerical analyses the introduced equity fund). Additionally, a guarantee is provided by purchasing a suitable option that covers for losses of the fund value at maturity. As in practice, e.g. within variable annuity contracts, to finance this option, an account proportional guarantee fee[7] g — typically quoted as annualized figure — is deducted from the client's account (additionally to the fees introduced above). This results in

$$A_{(t+\frac{1}{12})-} = A_t \cdot Perf_{t,t+\frac{1}{12}} \cdot (1-\gamma)^{\frac{1}{12}} \cdot (1-g)^{\frac{1}{12}} .$$

2.2.2. *Zero plus underlying.* This rather trivial but in many markets highly relevant product consists of a riskless asset (in our approach a zero-bond[8]) and a risky financial instrument (in our approach the above equity fund). Whenever new contributions enter the contract, the allocation in riskless and risky assets for the whole investment portfolio is determined as follows:

$$riskless_t = \min(A_t, F_t),$$

$$risky_t = A_t - riskless_t,$$

where $F_t := G_t \cdot \frac{p(t,T)}{(1-\gamma)^{T-t}}$ defines the so-called floor. Note, in the case $A_t < F_t$ this ensures that at most the currently available amount A_t is invested in the riskless asset and the product can be "underhedged", i.e. the product provider's loss is only realized at the contract's maturity.[9]

2.2.3. *Client-individual constant proportion portfolio insurance (iCPPI).* In this product, the well-known CPPI-algorithm[10] is applied on a client-individual basis. In theory, the asset allocation of CPPI-products is rebalanced continuously according to some given rule. In practice however, such re-allocations can only be applied at certain trading dates. At each such rebalancing time t (in our numerical analysis as typically also in practice: daily) the provider determines the asset allocation for each client's individual account by

$$risky_t = \max(0, \min(A_t, m \cdot (A_t - F_t))),$$
$$riskless_t = A_t - risky_t,$$

where m denotes the multiplier.[11] Hence, m times the so-called cushion $(A_t - F_t)$ is invested in the risky asset but (due to the borrowing constraint that is typically included in old-age provision contracts) no more than A_t.[12] CPPI products typically offered apply multipliers between three and five. Therefore, we use $m = 4$ throughout our numerical analyses in the following sections.

Obviously in practice — when continuous rebalancing is impossible — the product provider faces two sources of risk within a CPPI structure: first, the risky asset might lose more than $\frac{1}{m}$ during one period (this risk is often referred to as gap risk or overnight risk). Second, the floor might have changed within one period due to interest rate-fluctuations. Since we perform analyses from a client's perspective, we do not investigate how the product provider deals with these risks[13] and assume a flat additional charge on the risky asset within the iCPPI product instead.

2.3. Inflation-linked products

Our numerical analysis in Section 4 will show that the products considered so far, especially products with nominal guarantees, come with a significant downside risk in real terms. Therefore, in Section 5 we will introduce some modifications of the considered products that might be suitable to reduce inflation risk. Basically,[14] on the one hand we change the calculation of the floor F_t by taking into account the inflation accrued so far and some estimate for future inflation and on the other hand we analyze the impact of using inflation-linked zero-bonds instead of "standard" zero-bonds as risk-free asset.

3. Financial model

We start with an introduction of the (real-world) asset model used for our analysis where we add inflation risk to the model introduced in Graf *et al.* (2012) who consider a slightly modified version of the Heston model (cf. Heston, 1993) for

TABLE 1

CAPITAL MARKET PARAMETERS (WITHOUT INFLATION).

κ_r	θ_r	σ_r	$r(0)$	κ_V	θ_V	σ_V	$V(0)$	ρ_V	λ_S
20%	4.5%	7.5%	4.5%	475%	$(22\%)^2$	55%	$(22\%)^2$	−57%	3%

stock markets and the Cox-Ingersoll-Ross model (cf. Cox *et al.*, 1985) for interest rate markets. We add inflation by means of the Vasiçek model (cf. Vasiçek, 1977) which is typically used for modeling the term structure of interest rates. As already mentioned, this approach coincides with the economic scenario generator as introduced by Ahlgrim *et al.* (2005).

Let $(\Omega, \mathcal{F}, \mathfrak{F}, \mathcal{P})$ be a filtered probability space equipped with the natural filtration $\mathfrak{F} = (\mathcal{F}_t)_t = (\sigma(W^S(z), W^V(z), W^r(z), W^i(z), \ z \leq t))_t$ generated by (correlated) $\mathcal{P}-$Brownian Motions $W^S(t)$, $W^V(t)$, $W^r(t)$ and $W^i(t)$. Further, let $r(t)$ denote the (nominal) short-rate, $i(t)$ the annualized rate of inflation and $S(t)$ the equity's spot price at time t. The $\mathcal{P}-$dynamics of the asset model are then given by

$$
\begin{aligned}
dr(t) &= \kappa_r(\theta_r - r(t))\,dt + \sigma_r\sqrt{r(t)}\,dW^r(t),\\
di(t) &= \kappa_i(\theta_i - i(t))\,dt + \sigma_i\,dW^i(t),\\
dS(t) &= S(t)\,((r(t) + \lambda_S)\,dt + \sqrt{V(t)}\,dW^S(t)),\\
dV(t) &= \kappa_V(\theta_V - V(t))\,dt + \sigma_V\,dW^V(t),
\end{aligned}
$$

where λ_S denotes the equity risk premium. Further, the (instantaneous) correlation of the underlying Brownian Motions[15] is given as

$$
\Sigma = \begin{pmatrix} 1 & \rho_{ir} & \rho_{iS} & \rho_{iV} \\ \rho_{ri} & 1 & \rho_{rS} & \rho_{rV} \\ \rho_{Si} & \rho_{Sr} & 1 & \rho_{SV} \\ \rho_{Vi} & \rho_{Vr} & \rho_{VS} & 1 \end{pmatrix}.
$$

Within this setting, zero-bond prices $p(t, \ T+t)$ with time to maturity Tat time t are given by $p(t, \ T+t) = A(T)\exp(-B(T)r(t))$, with

$$
\begin{aligned}
A(T) &= \left[\frac{2\cdot h\cdot \exp(\kappa_r + h)\cdot \frac{T}{2}}{(\kappa_r + h)\cdot(\exp(h\cdot T) - 1) + 2\cdot h}\right]^{\frac{2\kappa_r\theta_r}{\sigma_r^2}},\\
B(T) &= \frac{2\cdot(\exp(h\cdot T) - 1)}{(\kappa_r + h)(\exp(h\cdot T) - 1) + 2\cdot h},
\end{aligned}
$$

where $h = \sqrt{\kappa_r^2 + 2\cdot\sigma_r^2}$.

For the sake of consistency, interest rate and equity parameters are directly adopted from Graf *et al.* (2012) and summarized in Table 1.

TABLE 2

INFLATION RATE PARAMETERS.

κ_i	θ_i	σ_i	$i(0)$
20%	2%	1%	2%

The parameters for the inflation model have been developed by applying a maximum likelihood approach, as e.g. proposed by Sørensen (1997) to data of the German CPI provided by the Deutsche Bundesbank.[16] The Index exists since the monetary reform in Germany in 1948 and is available for monthly and annual data. As already analyzed, e.g. in Ahlgrim *et al.* (2005), disruptions in monthly data might lead to overstated mean reversion strength and volatility parameters of the inflation rate process. Hence, we derive the parameters by considering annual averages of the respective CPI. Due to the rather high volatility of the annual CPI in the first years after World War II, we exclude these observations and concentrate on the time series reflecting the period from 1952 to 2010. Results of the maximum-likelihood estimates are summarized in Table 2.

The correlations were estimated using a discretization approach where we obtained $\rho_{ir} = \rho_{ri} = 33\%$ and $\rho_{iS} = \rho_{Si} = -15\%$. For consistency reasons with the results derived in Graf *et al.* (2012) we assume $\rho_{Sr} = \rho_{rS} = 0$. The variance process is assumed to be independent of the interest rate and the inflation rate.

We now define the CPI $I(t)$ at time t as $I(t) = e^{\int_0^t i(u)du}$ with $I(0) = 1$. Further, an inflation-linked zero-bond issued at time s with time to maturity T is defined as derivative on the inflation rate paying the relative increase $\frac{I(T)}{I(s)}$ of the underlying price index on a notional of 1. We denote its price at time t by $p_{I,s}(t, T)$.

In our numerical analyses, we approximate $p_{I,s}(t, T)$ using a Vasiçek-type interest rate process and the inflation rate process as introduced above. This allows for a quick pricing and hence for a broad quantitative simulation study outlined in more detail in Appendix A.

4. INFLATION ADJUSTED RISK-RETURN PROFILES OF THE STANDARD PRODUCTS

In this section we analyze the impact of inflation risk on risk-return profiles of the products introduced in Section 2. We analyze nominal and real internal rates of return (IRR) where real returns are defined as $IRR_{Real} = \frac{1+IRR_{Nominal}}{1+InflRate} - 1$, where $InflRate$ denotes the (annualized) inflation rate on the premiums invested.[17] Section 4.1 first treats the case of a single contribution to the products whereas Section 4.2 then gives some results for the case of regular contributions. All results were derived using Monte Carlo simulation techniques

TABLE 3

KEY FIGURES NOMINAL RETURNS — SINGLE PREMIUM.

	Nominal Zero-Bond (%)	Zero + Underlying (%)	iCPPI (%)	Option-Based Product (%)	Equity Fund (%)
5%	3.68	0.58	0.00	0.00	−4.08
25%	3.68	1.82	0.00	0.00	0.26
Median	3.68	3.39	2.07	2.80	3.23
75%	3.68	5.57	5.89	5.81	6.26
95%	3.68	9.29	10.30	10.07	10.54
Expected Return	3.68	5.31	5.61	5.57	5.93
P (IRR < 0%)	0.00	0.00	0.00	0.00	23.25
P (IRR < 2%)	0.00	28.20	49.69	42.89	39.14
Expected Shortfall	0.00	0.00	0.00	0.00	46.56
CTE 95	3.68	0.37	0.00	0.00	−5.63

generating 50,000 trajectories to derive estimates of the distributions of the considered products' rates of return.

Throughout the quantitative sections, we assume a term to maturity of $T = 30$ years and use $\beta = 5\%$, $\gamma = 0.5\%$ p.a., and $c = 1.3\%$ p.a. The guarantee fee applied within the static option-based product is taken from Graf *et al.* (2012) as $g = 0.43\%$ p.a. for a single premium and $g = 0.81\%$ p.a. for regular contributions. These values were derived by a risk-neutral valuation of the corresponding option. Further, for the iCPPI product we set $m = 4$ and apply an additional crash-protection fee of 0.2% p.a.

4.1. Single contribution

We start with analyzing a single contribution to the products introduced in Section 2. Figure 1 gives the empirical frequency distributions of internal rates of return.[18]

Next, Tables 3 and 4 summarize some key statistics derived from the empirical nominal and real returns, respectively.

First, it is worthwhile noting that due to the chosen modeling approach, estimated nominal returns of the considered products are similar to the nominal returns observed by Graf *et al.* (2012): the iCPPI product and the option-based product manage to combine guarantees with a potential for rather high returns. However, at the same time they show quite a lot of probability mass at a nominal return of 0% or slightly above. The zero plus underlying product shows a somewhat more balanced risk-return profile. The zero-bond investment does not show any randomness at all on a nominal basis.

When analyzing the same products' real returns, the impact of inflation risk becomes visible: the zero-bond investment (obviously) shows some volatility in real returns. While the nominal return was fixed at 3.68%, real returns deviate

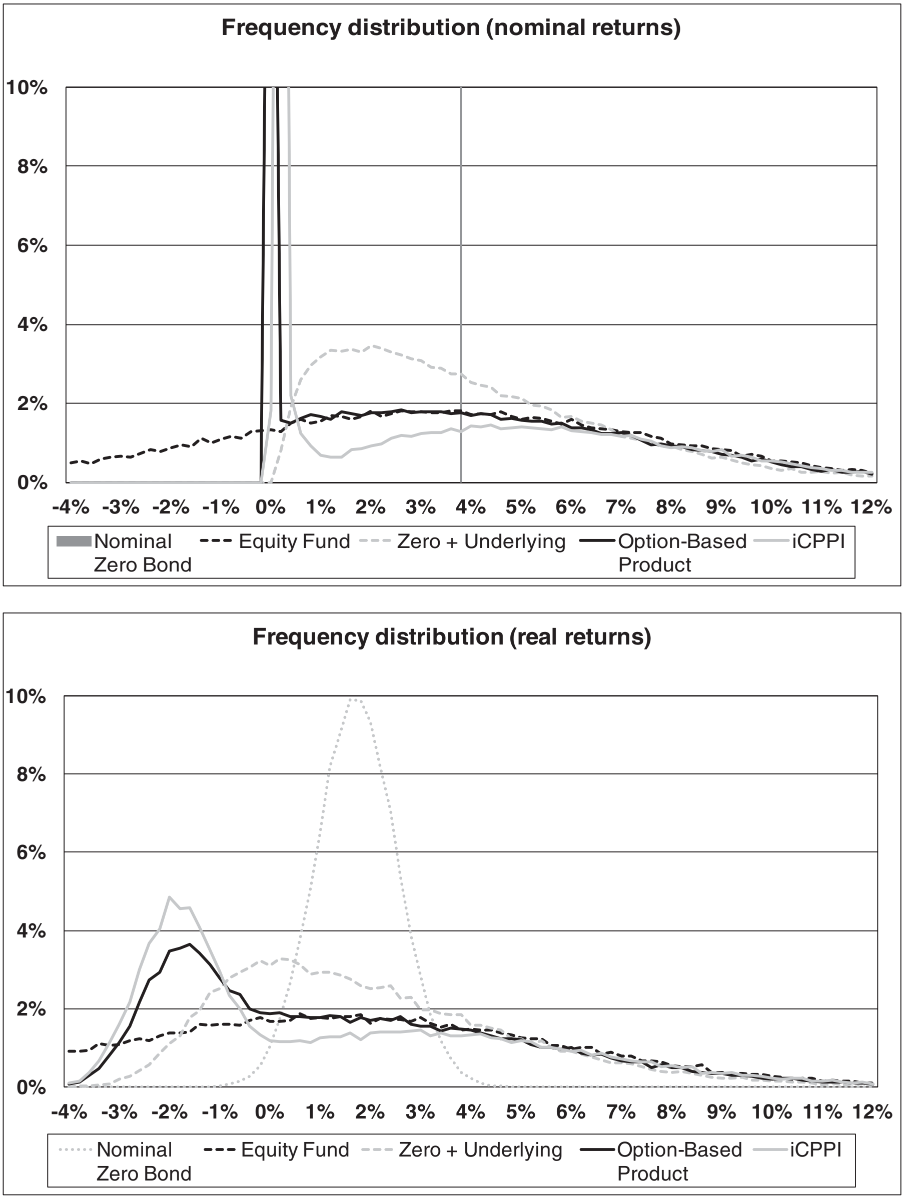


FIGURE 1: Empirical nominal (upper) and real (lower) rates of return — single contribution.

between 0.29% and 2.96% in 90% of the observed cases. With a probability of 2.1%, the real return even becomes negative.

An approximation often used in practice for assessing real rates of return is subtracting some assumed long-term average for the rate of inflation (e.g. 2% p.a.) from the nominal returns. Comparing the results in Tables 3 and 4, this

TABLE 4

KEY FIGURES REAL RETURNS — SINGLE PREMIUM.

	Nominal Zero-Bond (%)	Zero + Underlying (%)	iCPPI (%)	Option-Based Product (%)	Equity Fund (%)
5%	0.29	−1.79	−2.91	−2.75	−6.14
25%	1.08	−0.23	−1.86	−1.49	−1.80
Median	1.62	1.41	0.01	0.77	1.20
75%	2.16	3.59	3.85	3.78	4.22
95%	2.96	7.31	8.27	8.05	8.51
Expected Return	1.72	3.32	3.61	3.57	3.93
P (IRR < 0%)	2.10	28.66	49.93	42.98	39.48
Expected Shortfall	7.96	25.73	40.96	37.68	52.27
CTE 95	−0.01	−2.27	−3.26	−3.14	−7.69

approximation actually might deliver appropriate results when certain statistics such as expected returns are considered, but may massively fail when e.g. risk measures are analyzed. This can be seen by comparing e.g. the probability that nominal returns fall below 2% with the probability of a negative real return. Further, we will show in Section 5 that even for those statistics where the approximation might work well for the above products, the approximation completely fails when applied to products where the investment is linked to inflation.

Considering products with nominal investment guarantees, we find a significant exposure to inflation risk. The products with the highest nominal upside potential come with a significant point mass at a nominal rate of return of 0% p.a., i.e. generally losses in real terms. However, note that the empirical distribution of real rates of return does not yield a point mass at −2% p.a. which would be implied by the above approximation, but rather reveals the random characteristics of inflation.

Summarizing, considering inflation in risk-return profiles uncovers important characteristics of the considered products and especially challenges the perceived "safety" of products equipped with nominal guarantees.

4.2. Regular contributions

We also performed analyses for all considered products assuming a regular monthly premium payment, where premiums are level on a nominal basis. The difference in nominal risk-return profiles between products with single and regular premiums is explained in detail in Graf *et al.* (2012). For example, for the considered CPPI product, the difference in nominal risk-return profiles between the regular contributions and the single premium case is particularly large: new contributions help increasing the exposure to the risky asset even after the equity exposure has dropped to zero. This is possible because the new premium increases the (nominal) guarantee by this premium. The present value of the guarantee (i.e. the floor) is generally increased by less than the premium and a new "cushion" is created. This, of course, increases the return potential of the product. Consequently, the iCPPI's (nominal) median is higher when compared to the single premium case and — as opposed to the single premium case — even higher than that of the option-based product. These differences remain in place in the model framework of this paper and hence translate to similar results when inflation adjusted risk-return profiles are analyzed. Therefore, we refrain from a detailed description of the results.

However, note that investing in the considered fixed income asset on a regular basis results in a non-degenerated distribution of (both nominal and real) internal rates of return.

5. Inflation-Linked Product Designs

Section 4 showed that the considered products including products with nominal guarantees come with a significant downside risk in real terms. Hence, in this section we introduce possible product modifications that might be suitable to reduce inflation risk. Investing in an inflation-linked zero-bond as introduced in Section 3 may eliminate inflation risk if real interest rates exceed the product provider's charges, however does not provide any additional upside potential. Nevertheless, we include this type of investment in the following analyses.

Corrigan *et al.* (2011) propose ideas on issuing "variable annuity type" guarantees providing inflation risk protection by some form of an additional option on the mutual fund investment as introduced above. Further, they derive fair (i.e. risk-neutral) prices of these types of guarantees. Their results indicate that a complete option-based inflation risk protection may come at a rather high cost. Hence, full coverage of inflation risk by means of an option seems very expensive resulting in products with rather limited upside potential. To the best of our knowledge no research has been done with respect to modifying the other products introduced above in order to reduce or eliminate inflation risk. However, Fulli-Lemaire (2012) introduces some hybrid trading strategy taking into account inflation estimates and analyzes their potential using historical back-testing and block bootstrapping techniques.

In what follows, we focus on modifying the iCPPI and the zero plus underlying products in order to provide some reduction (not necessarily a complete elimination) of inflation risk. Our modifications are based on two basic ideas: (1) incorporating some estimate for future inflation in the guarantee basis which then impacts the calculation of the floor F_t and hence results in a different asset allocation when compared to the original products and (2) using a different asset than nominal zero-bonds as a "risk-free asset", e.g. an inflation-linked zero-bond, and calculate the floor accordingly (see below). Similar to Section 2, we state the necessary formulae for both single and regular contributions, however concentrate on quantitative analyses of a single contribution and only briefly comment on the results for regular contributions where additional insights can be gained.

Note that vis-a-vis the client, the modified products come *without* any nominal or real investment guarantee. Nevertheless, we still use the terms "guarantee basis" and "floor" as introduced in Section 2.

5.1. Modification of the nominal guarantee basis G_t

Our first approach is a modification of the nominal guarantee basis G_t by allowing for inflation. At time t the modified guarantee basis G_t consists of the realized inflation on the premiums until time t (which is a $\mathcal{F}_t$-measurable random variable determined by $I(t)$ as introduced in Section 3 and some estimate $\tilde{i}(t)$ for the future rate of inflation within $(t, T]$.

We set

$$G_t = P \cdot \frac{I(t)}{I(0)} \cdot (1 + \tilde{i}(t))^{T-t}$$

when a single premium is considered and

$$G_t = \begin{cases} \left(\sum_{i=0}^{[t \cdot 12]} P \cdot \frac{I(t)}{I\left(\frac{i}{12}\right)} \right) (1 + \tilde{i}(t))^{T-t}, & t < T \\ \sum_{i=0}^{T \cdot 12 - 1} P \cdot \frac{I(T)}{I\left(\frac{i}{12}\right)}, & t = T \end{cases}$$

when regular contributions are in place.

We consider two different approaches for estimating $\tilde{i}(t)$: in the *historic* approach, we use the realized rate of inflation until time t as an estimate for the future rate of inflation as well. In the *market* approach, we use the market's expectation on the future rate of inflation until maturity T.

In the historic approach we obviously obtain $\tilde{i}(t) = (\frac{I(t)}{I(0)})^{\frac{1}{t}} - 1$ for the single premium case and $\tilde{i}(t) = IRR((P, \ldots, P), \sum_{i=0}^{[t \cdot 12]} P \cdot \frac{I(t)}{I(\frac{i}{12})})$ in the regular premium case where $IRR(\mathrm{x}, y)$ gives the IRR of a vector of contributions x and a corresponding benefit y.

In the market approach, we instead use the market's expectation for the future rate of inflation implied by inflation-linked derivatives (e.g. inflation-linked bonds or inflation swaps). According to Corrigan *et al.* (2011) and e.g. a report by Kerkhof (2005), the market for zero-coupon inflation swaps is amongst the most liquid inflation-linked derivatives markets. In our model setup, the inflation swap issued at time t delivers the relative increase $\frac{I(T)}{I(t)}$ at maturity T against the fixed rate K on a notional of 1. No arbitrage arguments then yield (cf. e.g. Mercurio, 2005) the fair swap rate K to fulfill $(1 + K)^{T-t} = \frac{p_{I,t}(t,T)}{p(t,T)}$.[19] Hence, the fair swap rate is (without any model assumption) uniquely determined by the prices of nominal and inflation-linked zero-bonds. Further, note that the swap rate K in general not only shows the market's expectation for the future rate of inflation but also includes various additional risk premiums such as e.g.

liquidity or credit risk. Hence, these additional risk premiums may be filtered out to obtain a "best estimate" of the market's inflation expectation as e.g. done in Schulz and Stapf (2009). Since we do not model any additional risk factors, we extract the market's inflation expectation directly from K and set $\tilde{i}(t) = K$.[20]

Of course, the historic approach is easier implemented in practice, since no additional data are required. However, if significant changes in the (future) rate of inflation occur, e.g. due to a change in the monetary policy, the historic approach only reacts with a certain time-lag whereas the market approach should be able to pick up this change rather quickly.

Based on the modified guarantee basis, the allocation mechanisms for the iCPPI and the zero plus underlying remain unchanged as described in Section 2. In particular, in this approach, the products still use nominal zero-bonds as riskless assets.

5.2. Use of inflation-linked zero-bonds instead of nominal zero-bonds

Neglecting liquidity issues that may arise in a practical application, in our second approach, we use inflation-linked bonds as "safe assets" in the iCPPI and zero plus underlying products introduced in Section 2. For ease of notation, we assume that the product provider solely invests in an inflation-linked zero-bond issued at time 0.

Similar with Section 2, let G_t denote the guarantee base at time t. However, here we identify G_t as the number of units that have to be invested in the considered inflation-linked zero-bond in order to provide the inflation protection for the invested premium(s). If a premium P is contributed to the contract at time t, $P \cdot \frac{I(0)}{I(t)}$ units of the above inflation-linked zero-bond deliver $(P \cdot \frac{I(0)}{I(t)})(\frac{I(T)}{I(0)} \cdot (1-\gamma)^{T-t}) = P \cdot \frac{I(T)}{I(t)} \cdot (1-\gamma)^{T-t}$, which exactly gives the premium P after adjusting for inflation. Hence, we set $G_0 = G_t = P \, \forall t$ for single premium contracts. For regular premium contracts, we set $G_0 = P$ and $G_t = G_{t-\frac{1}{12}} + P \cdot \frac{I(0)}{I(t)}$ at every monthly premium payment date t. The floor F_t is consequently calculated as $F_t = G_t \cdot \frac{p_{I,0}(t,T)}{(1-\gamma)^{T-t}}$. The asset allocation of the zero plus underlying and the iCPPI products then follow exactly the formulae introduced in Section 2, but using the floor F_t introduced in this section and the inflation-linked zero-bond as riskless asset.

Note, that the product introduced in Section 5.2 is somehow similar to the product introduced in Section 5.1 applying a market-based approach, since this product's floor can be rewritten as

$$G_t \cdot \frac{p(t,T)}{(1-\gamma)^{T-t}} = P \cdot \frac{I(t)}{I(0)}(1+\tilde{i}(t))^{T-t} \cdot \frac{p(t,T)}{(1-\gamma)^{T-t}}$$

$$= P \cdot \frac{I(t)}{I(0)} \cdot \frac{p_{I,t}(t,T)}{p(t,T)} \cdot \frac{p(t,T)}{(1-\gamma)^{T-t}}$$

$$= P \cdot \frac{I(t)}{I(0)} \cdot p_{I,t}(t, T) \cdot \frac{1}{(1-\gamma)^{T-t}}$$

$$= P \cdot p_{I,0}(t, T) \cdot \frac{1}{(1-\gamma)^{T-t}} = G_t \cdot \frac{p_{I,0}(t, T)}{(1-\gamma)^{T-t}},$$

which coincides with the floor from the product introduced in this section when a single contribution is considered. Similar derivations are possible for regular contributions. Hence, both products allocate the same amount of (nominal) money into the riskless asset but assume a different riskless asset.

5.3. Results for single premium contracts

Figures 2 and 3 show the frequency distribution of the return of the modified products on a nominal and real basis, respectively.

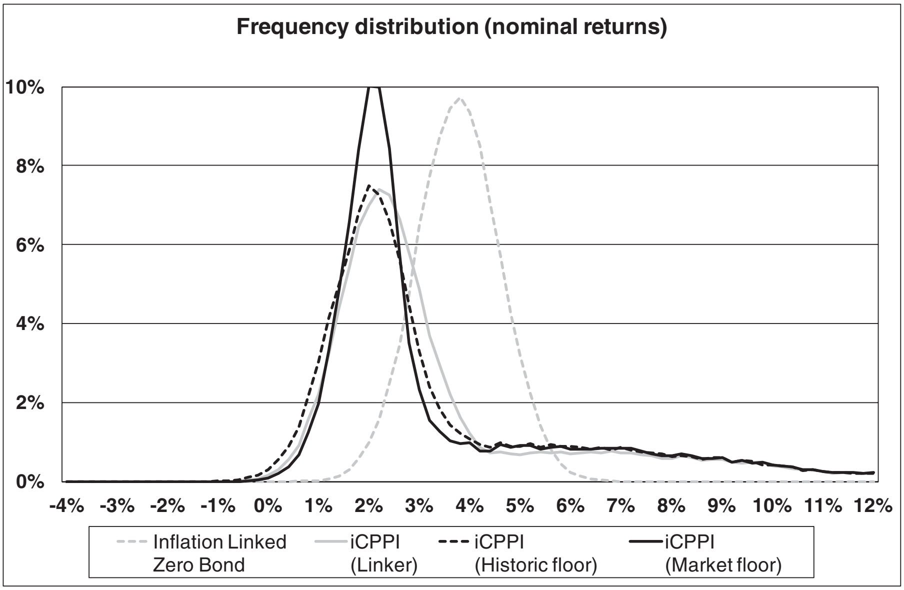
Further, Tables 5 and 6 summarize some key statistics of these empirical distributions.

First, note that the approximation often used by practitioners to assess real returns by means of nominal returns less some estimate for the long-term rate of inflation now completely fails as an approximation for the probability distribution of real returns. While for the products analyzed in Section 4 the approximation delivered appropriate results at least for expected values, a deterministic correction for inflation is not suitable at all for path-dependent inflation-linked products.

An analogy between the inflation-linked and the nominal zero-bond is identified by observing a degenerated probability distribution in real terms and a non-degenerated probability distribution in nominal terms for the inflation-linked zero-bond returns and vice versa for the nominal zero-bond.

Now, we have a closer look at the products resulting from the first approach, i.e. the products with a modified guarantee basis (where still nominal zero-bonds are used as a risk-free asset). These products are denoted by "historic floor" and "market floor" (depending on the used estimate $\tilde{i}(t)$) in the figures and tables. Compared to the products with nominal guarantees from Section 4, the probability and the extent of negative real returns are reduced significantly (cf. Tables 4 and 6), however real losses are still possible. This is mainly due to the fact that the "guarantee" G_t may not be sufficient whenever realized inflation deviates from estimated inflation.

Comparing the historic and the market approaches, we find that the market approach may deliver better results than the historic approach especially when lower percentiles of the real returns' distributions are used as a risk measure. Hence, the extent of real losses (if they occur) is generally lower when a market-based approach is implemented instead of a pure historic estimate, since the market-based approach generally picks up changes in the future rate of inflation more quickly than the historic approach does.[21]



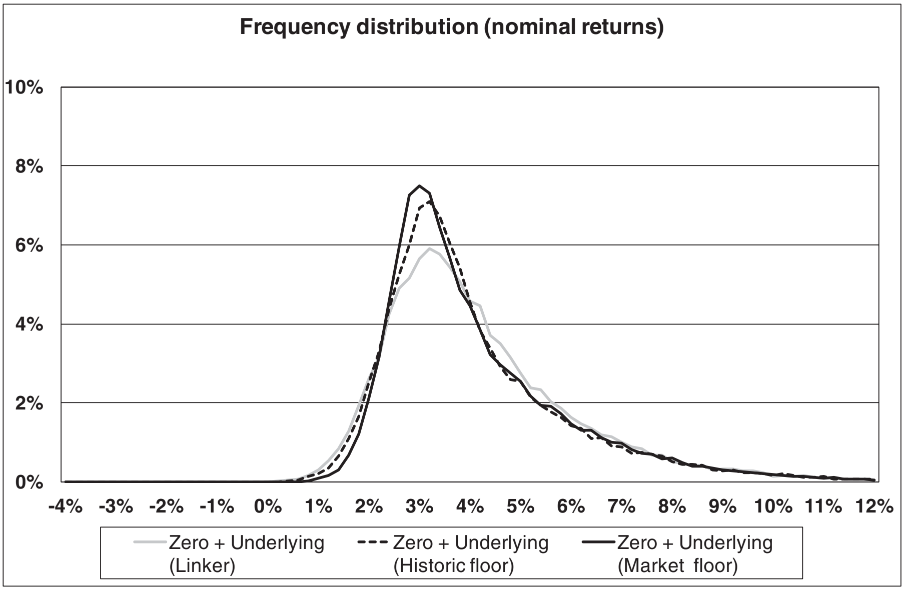


FIGURE 2: Empirical nominal rates of return — iCPPI/inflation-linked zero (upper) and zero plus underlying (lower).

We now look at the second approach for modification, i.e. products where inflation-linked zero-bonds instead of nominal zero-bonds are used as a safe asset. These two products are denoted by "linker" in the figures and tables. Although nominal losses are possible (cf. Table 5),the risk of negative real returns

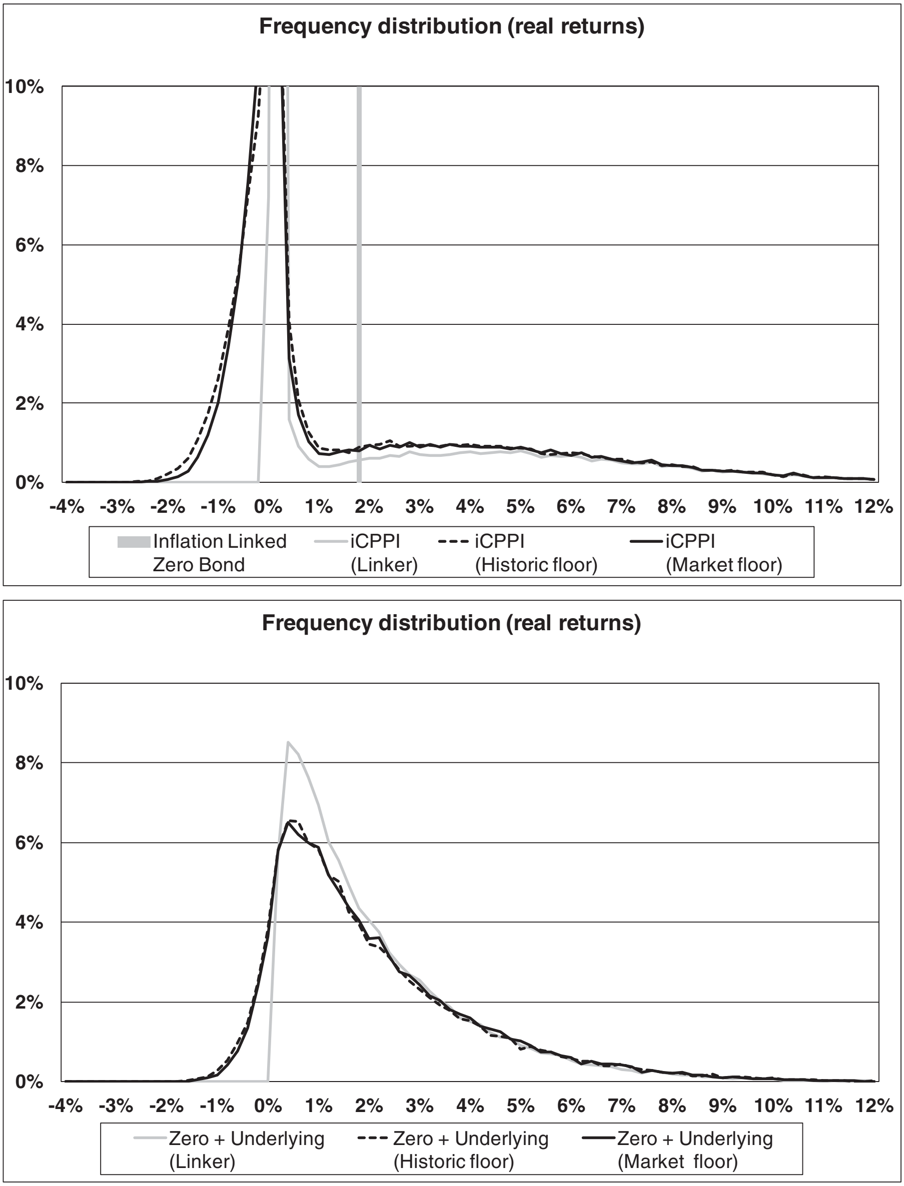


FIGURE 3: Empirical real rates of return — iCPPI / inflation-linked zero (upper) and zero plus underlying (lower).

is significantly reduced: considering the zero plus underlying product, there is no risk of negative real returns in the single premium case[22] whereas for the iCPPI product, real losses are possible due to gap events which are however not observed in our simulation study. Further, the probability distributions of real rates for the modified products and the corresponding probability distributions

TABLE 5

KEY FIGURES NOMINAL RETURNS — SINGLE PREMIUM (REDESIGNED PRODUCTS).

	Inflation Linked Zero-Bond (%)	Zero + Underlying (Historic Floor) (%)	iCPPI (Historic Floor) (%)	Zero + Underlying (Market Floor) (%)	iCPPI (Market Floor) (%)	Zero + Underlying (Linker) (%)	iCPPI (Linker) (%)
5%	2.32	1.87	0.75	2.03	1.03	1.78	0.93
25%	3.11	2.78	1.64	2.78	1.74	2.79	1.79
Median	3.66	3.51	2.34	3.51	2.25	3.67	2.49
75%	4.22	4.78	4.42	4.83	4.38	4.97	3.83
95%	5.04	7.83	9.78	7.82	9.83	7.82	9.77
Expected Return	3.76	4.67	5.21	4.68	5.23	4.74	5.18
P (IRR < 0%)	0.00	0.01	0.62	0.00	0.15	0.01	0.27
P (IRR < 2%)	2.09	6.77	37.95	4.66	37.74	7.85	32.31
Expected Shortfall	0.00	4.06	7.63	0.00	5.51	4.59	6.85
CTE 95	1.99	1.51	0.40	1.73	0.72	1.41	0.60

TABLE 6

KEY FIGURES REAL RETURNS — SINGLE PREMIUM (REDESIGNED PRODUCTS).

	Inflation Linked Zero-Bond (%)	Zero + Underlying (Historic Floor) (%)	iCPPI (Historic Floor) (%)	Zero + Underlying (Market Floor) (%)	iCPPI (Market Floor) (%)	Zero + Underlying (Linker) (%)	iCPPI (Linker) (%)
5%	1.61	−0.27	−1.13	−0.22	−0.95	0.18	0.00
25%	1.61	0.49	−0.36	0.52	−0.31	0.67	0.00
Median	1.61	1.37	0.03	1.43	0.01	1.46	0.00
75%	1.61	2.87	2.60	2.93	2.61	2.84	1.59
95%	1.61	6.12	7.88	6.05	7.89	5.79	7.81
Expected Return	1.61	2.75	3.26	2.74	3.27	2.67	3.17
P (IRR $<$ 0%)	0.00	9.80	45.89	8.91	48.37	0.00	0.00
P (IRR $<$ 0.01%)	0.00	10.04	47.14	9.19	50.44	0.02	60.76
Expected Shortfall	0.00	10.03	13.81	9.33	11.51	0.00	0.00
CTE 95	1.61	−0.59	−1.46	−0.52	−1.24	0.11	0.00

TABLE 7

KEY FIGURES REAL RETURNS — REGULAR PREMIUM (REDESIGNED PRODUCTS).

	Zero + Underlying (Linker) (%)	iCPPI (Linker) (%)
5%	0.23	−0.04
25%	0.73	−0.01
Median	1.44	0.05
75%	2.67	2.01
95%	5.46	8.50
Expected Return	2.33	2.95
P (IRR < 0%)	0.07	31.43
P (IRR < 0.01%)	0.12	39.02
Expected Shortfall	0.33	0.33
CTE 95	0.14	−0.06

of nominal rates for the "standard" products (cf. Figures 1 and 2) have a similar shape.

Both products still offer some upside potential. Clearly, the iCPPI product comes with more fluctuation compared to the zero plus underlying product. In the 95th percentile, for example, the iCPPI versions even under real returns achieve an IRR of almost 8%. However, the cost of providing a very high upside potential in good capital market scenarios goes hand in hand with a very high probability of approximately 60% of "just" getting the inflated premiums back. In contrast, the zero plus underlying delivers a more moderate distribution of real returns, which results in less upside potential but also a lower probability of very low or even zero real returns. Hence, considering these products and products "in between these two products" seems a promising way of creating different inflation-protected products for clients with different degrees of risk aversion.

Finally note that, although in our simulation study no negative real returns were observed, there is no guarantee embedded in the products. If e.g. a massive increase in the expected future rate of inflation (and thus massive increase in the inflation-linked zero-bonds' price) occurs simultaneously with a dramatic loss in equity, especially the introduced iCPPI product may not be able to buy the required inflation-linked zero-bonds after that event and hence real losses are possible.

5.4. Regular contributions

We performed the same analyses for regular premium payments and will provide a summary of the most interesting results. The zero plus underlying and the iCPPI product based on a historic floor or market floor don't show any additional (i.e. non-expected) effects in the case of regular premiums payments and we hence refrain from a detailed discussion of the corresponding results.

The products using an inflation-linked bond, however, show some interesting additional effects that are displayed in Table 7.

When regular contributions are considered, the products also try to provide protection against inflation for future contributions. Even if the products are able to completely eliminate inflation risk in the single premium case (cf. Section 4.1) this may not be possible in the regular premium case (e.g. due to possibly lower real interest rates in the future). Hence, in contrast to the single premium case both products now come with a positive probability of negative real returns. While for the zero plus underlying product this is still rather unlikely, a negative real return is observed in about one-third of the cases for the iCPPI product. The extent of negative real returns, however, is very small in both cases. This, for example, can be seen by comparing the CTE 95 or the expected shortfall. Thus, even though a guarantee of non-negative real returns is not possible, the two products provide a rather effective protection against inflation also in the regular premium case.

6. Sensitivity Analyses

In this section we analyze sensitivities of our results with respect to capital market assumptions for the products analyzed in Sections 4 and 5. In particular, we vary the underlying correlation structure and the level of nominal and model-induced real interest rates. We concentrate on single premium contracts and real returns. Note that, within this section we only comment on the most important effects and show a comprehensive set of sensitivity results in Appendix B.

6.1. Sensitivity with respect to the correlation structure

At first we stress the assumed correlation matrix of the considered asset classes' random innovations. The correlation matrix used in the previous analyses was

$$\Sigma = \begin{pmatrix} 1 & \rho_{ir} & \rho_{iS} & \rho_{iV} \\ \rho_{ri} & 1 & \rho_{rS} & \rho_{rV} \\ \rho_{Si} & \rho_{Sr} & 1 & \rho_{SV} \\ \rho_{Vi} & \rho_{Vr} & \rho_{VS} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 33\% & -15\% & 0 \\ 33\% & 1 & 0 & 0 \\ -15\% & 0 & 1 & -57\% \\ 0 & 0 & -57\% & 1 \end{pmatrix}.$$

Sensitivity I

We first assume all correlations are zero except for the correlation between stock returns and their volatility, i.e.

$$\Sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -57\% \\ 0 & 0 & -57\% & 1 \end{pmatrix}.$$

Sensitivity II

Next, we analyze the effect of extreme values of ρ_{ri} by setting them to $\pm 100\%$. Since the resulting "correlation matrix" may not be positive semi-definite by construction, we apply the approach introduced by Higham (2002) and adjust the correlation matrix accordingly.

Hence, sensitivity II(a) assuming an extreme positive correlation results in the following correlation matrix:

$$\Sigma = \begin{pmatrix} 1 & 98.50\% & -14.68\% & 0.18\% \\ 98.50\% & 1 & -0.31\% & -0.17\% \\ -14.68\% & -0.31\% & 1 & -56.96\% \\ 0.18\% & -0.17\% & -56.96\% & 1 \end{pmatrix}.$$

Sensitivity II(b) assuming an extreme negative correlation is accordingly summarized by

$$\Sigma = \begin{pmatrix} 1 & -98.50\% & -14.68\% & 0.18\% \\ -98.50\% & 1 & 0.31\% & 0.17\% \\ -14.68\% & 0.31\% & 1 & -56.96\% \\ 0.18\% & 0.17\% & -56.96\% & 1 \end{pmatrix}.$$

Results:

First note that the distribution of real investment returns remains unchanged for the nominal zero-bond.[23] Further, Table B1 in Appendix B shows that the assumption of a zero correlation does not change the results too much and hence we only comment on the extreme positive and negative correlation assumptions.

The equity fund investment and the option-based product provide a pure investment in the considered equity process whose performance is influenced by the interest rates and hence due to the correlation also related to the rate of inflation. We observe less variability when a positive correlation of interest rates and the rate of inflation is assumed: the considered probability distributions' lower respectively upper tails (i.e. the 5th and 95th percentile) increase and decrease respectively. For a negative correlation, we observe the opposite effect. A similar effect occurs with the considered CPPI products, on the one side due to the same effect in the equity part and on the other side due to the fact that when the inflation rate increases, the product's floor — as a function of the considered interest rates — potentially decreases (increases) when correlation is positive (negative) (cf. Table B2) leaving the products with higher (lower) equity share.

For the modified products as introduced in Section 5, it is worthwhile noting that, although the payout of the inflation-linked zero-bond does not change, we observe a different deterministic real return of the considered investment product when compared to the base-case. This is due to the inflation-linked bond's different price at $t = 0$. Further, the modified products are generally more volatile when negative correlations of interest rates and inflation rates are assumed, leaving the products with a potentially higher upside at the cost of a more pronounced downside as well (cf. Table B2).

6.2. Sensitivity with respect to the level of nominal interest rates and rate of inflation

We now stress the level of nominal interest rates and inflation. Again, all other assumptions remain unchanged, in particular the "base case" correlation matrix as introduced in Section 3 is applied.

Sensitivity III

We assume an increase (decrease) of both, the start level and the long-term average of interest rates and inflation, i.e. $\theta_r = r(0) = 5.5\%$, $\theta_i = i(0) = 3\%$ in sensitivity III(a) and $\theta_r = r(0) = 3.5\%$, $\theta_i = i(0) = 1\%$ in sensitivity III(b).

Results:

Since in this sensitivity analysis, the level of interest rates and inflation is stressed by the same amount, the level of real rates remains more or less unchanged. Therefore, the real probability distributions of pure equity and fixed income investment are very similar to the base case. However, when considering CPPI products, this observation changes tremendously. When the level of interest rates is increased (decreased), the equity exposure of the considered products increases (decreases) as well which generally generates more (less) volatility in the products. Further, the nominal guarantee's value in real terms decreases (increases) when the level of inflation increases (decreases). These effects therefore increase the product's downside (in terms of real returns) even when interest rates are higher (cf. Table B3).

In contrast, the modified products as introduced in Section 5 are generally only influenced by the level of real interest rates and hence, their probability distributions remain largely unchanged when sensitivity III is applied (cf. Table B3) since real rates stay at a similar level.

Sensitivity IV

Finally we investigate an increase (decrease) of the level of real interest rates by 1% by assuming $\theta_r = r(0) = 5.5\%$ in sensitivity IV(a) and $\theta_r = r(0) = 3.5\%$ in sensitivity IV(b) while leaving $\theta_i = i(0) = 2\%$ unchanged.

Results:

When real rates increase (decrease) the effects on the standard products are exactly as expected: we observe a positive (negative) shift of the resulting return distributions (cf. Table B4). Increasing the level of real interest rates decreases the price of an inflation-linked zero-bond and hence, increases the cushion of all considered modified guarantee products (and vice versa). Therefore, the results for these products are (obviously) worse when real interest rates tend to be low. Additionally, path-dependant products suffer more than path-independent products since the former essentially result in very skewed return distributions (cf. Table B4).

7. Conclusion and Outlook

This paper analyzed the impact of inflation risk on the risk-return profiles of various old-age provision products. After extending the model introduced in

Graf *et al.* (2012) by taking inflation risk into account we derived risk-return profiles for various common standard old-age provision products assuming e.g. different portfolio insurance strategies. We found that most products — including products that are often considered particularly safe by practitioners and regulators due to nominal guarantees — bear a significant inflation risk. Therefore, information about a product's return distribution in real terms (that is not revealed by information provided to clients and their financial advisors so far) is relevant for sustainable financial planning. Finally, we have proposed product modifications that may reduce or even eliminate the risk of negative real investment returns while still allowing for some upside potential.

Of course, this work is only a starting point in this area of research. Future research could include an assessment of model risk, in particular taking into account the impact of the recent monetary policy by the various central banks due to the debt crisis. In particular it would be interesting to investigate the risk-return profile of the above products under an asset model allowing for lagged effects between the different asset classes. Further, it seems worthwhile studying whether the inflation-linked derivatives market is liquid enough for institutional investors to actually implement the proposed product modifications in practice.

One more field of future research will be an analysis of different retirement products in the payout phase. Here, a comparison of traditional annuities with innovative unit-linked product versions — which currently still play only a minor role in many countries — seems worthwhile. In contrast to the analyses of a one-time maturity benefit only, analyses of payout products have to deal with a stream of (life-contingent) benefits.

Notes

1. Note, however, that — whilst this law (called "Altersvorsorge-Verbesserungsgesetz") has been passed — details about the risk-return classes, in particular the stochastic model to be used, have not been decided on. The law states that an authority will be founded which will specify the rules.
2. See e.g. Beletski and Korn (2006).
3. The base contracts described in this section are similar to those analyzed in Graf *et al.* (2012) which contains more details on the modeling approach.
4. Cf. e.g. Section 2.3.1 in Bauer *et al.* (2008).
5. Cf. e.g. Section 2 in Kochanski and Karnarski (2011).
6. Note, it is very common in the old-age provision market to offer products with 100% guarantee of the contributions made. Hence, we do not consider products with different guarantee levels although such designs could analogously be considered in our framework.
7. g is fixed at the outset and not adjusted later on. The product provider typically invests the guarantee fee in some derivative security (or hedge portfolio) on the considered fund to hedge the guarantee. Since we focus on analyses from a client's perspective, this is not further investigated in this paper.
8. We ignore default risk in our model.
9. $F_t > A_t$ is possible due to charges and (in the iCPPI product introduced below) also due to fluctuations in interest rates and equity.
10. Cf. Black and Perold (1992).
11. Obviously, for a multiplier of 1, the zero plus underlying product and the iCPPI product coincide.
12. For an analytical treatment of CPPI strategies without borrowing constraint compare, e.g. Black and Perold (1992). As a consequence of the borrowing constraint, the analytical tractability

is lost. We still chose to analyze products with a borrowing constraint since these are predominantly offered and products without a borrowing constraint may behave significantly different.

13. Cf. Graf *et al.* (2012) and references therein for more details.

14. Details and respective formulae are provided in Section 5.

15. Note, that the processes $r(t)$, $s(t)$ and $i(t)$ are potentially correlated differently than their Brownian motions.

16. The considered time series can be found under: http://www.bundesbank.de/statistik/%20statistik_zeitreihen.php and signature UJFB99.

17. This definition is in line with most academic papers treating inflation risk and is e.g. introduced in Ibbotson and Sinquefield (1976; an academic outlet of the Ibbotson investment reports).

18. The internal rate of return of the considered cash flow is calculated as $IRR = (\frac{A_T}{P})^{\frac{1}{T}} - 1$ for a single contribution and as the solution of $A_T = P\sum_{t=0}^{12\cdot T-1}(1+IRR)^{T-\frac{t}{12}}$ for the regular contribution case.

19. Note, the swap rate K is usually directly quoted by brokers, e.g. on Bloomberg.

20. Within the Appendix we show how prices $p_{I,t}(t, T)$ of the inflation-linked zero-bond are computed in our work and how we are hence able to derive an estimate for K.

21. In a model with "regime switches" in the monetary policy this effect might even be more pronounced.

22. Obviously this only holds if $A_0 \geq F_0$.

23. Note, this effect is only observed for a single contribution and changes when regular contributions are in place.

REFERENCES

AHLGRIM, K.C., D'ARCY, S.P. and GORVETT, R.W. (2005) Modeling financial scenarios: A framework for actuarial profession. *Proceedings of the Casualty Actuarial Society*, **92**, 177–238.

AMENC, N., MARTELLINI, L. and ZIEMANN, V. (2009) Alternative investments for institutional investors, risk budgeting techniques in asset management and asset-liability management. *The Journal of Portfolio Management*, **35**(4), 94–110.

BAUER, D., KLING, A. and RUß, J. (2008) A universal pricing framework for guaranteed minimum benefits in variable annuities. *ASTIN Bulletin*, **38**(2), 621–651.

BELETSKI, T. and KORN, R. (2006) Optimal investment with inflation-linked products. In *Advances in Risk Management* (ed. G.N. Gregoriou), 170–190. Hampshire, UK: Palgrave-Mac Millan.

BELGRADE, N., BENHAMOU, E. and KOEHLER, E. (2004) A market model for inflation. *CDC Ixis Capital Markets*. Available at: SSRN http://ssrn.com/abstract=576081.

BLACK, F. and PEROLD, A.F. (1992) Theory of constant proportion portfolio. *Journal of Economic Dynamics and Control*, **16**(3–4), 403–426.

BRIÈRE, M. and SIGNORI, O. (2012) Inflation-hedging portfolios: Economic regimes matter. *The Journal of Portfolio Management*, **38**(4), 43–58.

CORRIGAN, J., DE WEIRDT, M., FANG, F. and LOCKWODD, D. (2011) Manufacturing Inflation Risk Protection. *Institute of Actuaries of Australia*. Available at: http://www.actuaries.asn.au/library/events/Conventions/2011/Con2011_Paper_Corrigan.pdf

COX, J.C., INGERSOLL, J.E. and ROSS, S.A. (1985) A theory of the term structure of interest rates. *Econometrica*, **53**(2), 385–407.

FULLI-LEMAIRE, N. (2012) *A Dynamic Inflation Hedging Trading Strategy Using a CPPI*. Available at SSRN: http://ssrn.com/abstract=1978438.

GRAF, S., KLING, A. and RUß, J. (2012) Financial planning and risk-return profiles. *European Actuarial Journal*, **2**(1), 77–104.

HESTON, S.L. (1993) A closed-form solution for options with stochastic volatility with applications to bond and currency options. *The Review of Financial Studies*, **6**(2), 327–343.

HIGHAM, N.J. (2002) Computing the nearest correlation matrix – a problem from finance. *IMA Journal of Numerical Analysis*, **22**, 329–343.

IBBOTSON, R.G. and SINQUEFIELD, R.A. (1976) Stocks, bonds, bills, and inflation: Year-by-Year historical returns (1926–1974). *The Journal of Business*, **49**(1), 11–47.

JARROW, R. and YILDIRIM, Y. (2003) Pricing treasury inflation protected securities and related derivatives using an HJM Model. *Journal of Financial and Quantitative Analysis*, **38**(2), 337–358.

KERKHOF, J. (2005) Inflation derivatives explained markets, products, and pricing. *Fixed Income Quantitative Research Lehmann Brothers*. Available at http://www.scribd.com/doc/86770535/19601807-Lehman-Brothers-Kerkhof-Inflation-Derivatives-Explained-Markets-Products-and-Pricing.

KOCHANSKI, M. and KARNARSKI, B. (2011) Solvency capital requirement for hybrid products. *European Actuarial Journal*, **1**(2), 173–198.

MERCURIO, F. (2005) Pricing inflation-indexed derivatives. *Quantitative Finance*, **5**(3), 289–301.

SCHULZ, A. and STAPF, J. (2009) Price discovery on traded inflation expectations: Does the financial crisis matter? *Deutsche Bundesbank Disussion Paper Series 1: Economic Studies No 25/2009*.

SØRENSEN, M. (1997) Estimating functions for discretely observed diffusions: A review. *Lecture Notes-Monograph Series*, **32**, 305–325.

VASIÇEK, O. (1977) An equilibrium characterization of the term structure. *Journal of Financial Economics*, **5**, 177–188.

WEIYIN, F. (2013) Optimal consumption and portfolio under inflation and Markovian switching. *Stochastics An International Journal of Probability*, **85**(2), 272–285.

STEFAN GRAF (Corresponding author)
Institut für Finanz- und Aktuarwissenschaften (ifa), Ulm, Germany
E-Mail: s.graf@ifa-ulm.de

LENA HAERTEL
Ulm University, Ulm, Germany
E-Mail: lena.haertel@uni-ulm.de

ALEXANDER KLING
Institut für Finanz- und Aktuarwissenschaften (ifa), Ulm, Germany
E-Mail: a.kling@ifa-ulm.de

JOCHEN RUß
Institut für Finanz- und Aktuarwissenschaften (ifa) and Ulm University, Ulm, Germany
E-Mail: j.russ@ifa-ulm.de

APPENDIX

A. PRICING OF INFLATION-LINKED ZERO-BONDS

In this appendix, we briefly sketch the implemented pricing of inflation-linked zero-bonds used in the products in Section 5. Note that in the considered asset model introduced in Section 3 nominal interest rates follow a Cox-Ingersoll-Ross model and rates of inflation follow a Vasiçek model. To the best of our knowledge, no closed form solution for pricing inflation-linked zero-bonds is available in this setup. Hence, we assume the following approximation for pricing: We assume the rate of inflation and the nominal interest rates to follow correlated Vasiçek processes:

$$dr(t) = \tilde{\kappa}_r(\tilde{\theta}_r - r(t))dt + \tilde{\sigma}_r dW^r(t),$$

$$di(t) = \kappa_i(\theta_i - i(t))\,dt + \sigma_i dW^i(t),$$

with $dW^i(t)dW^r(t) = \rho_{ri}$. This model is "consistent" with the original model (cf. Section 3) if we set $\tilde{\kappa}_r = \kappa_r$, $\tilde{\theta}_r = \theta_r$ and $\tilde{\sigma}_r = \sigma_r\sqrt{\theta_r}$. By no-arbitrage arguments, the price at time t of an inflation-linked zero-bond $p_{I,s}(t,T)$ issued at time $s(s \le t)$ with time-to-maturity T is then derived as

$$\begin{aligned} p_{I,s}(t,T) &= \mathbb{E}_Q\left[\exp\left(-\int_t^T r(u)\,du\right)\exp\left(\int_s^T i(u)\,du\right)\middle|\mathcal{F}_t\right] \\ &= \exp\left(\int_s^t i(u)\,du\right)\mathbb{E}_Q\left[\exp\left(-\int_t^T r(u)du\right)\exp\left(\int_t^T i(u)\,du\right)\middle|\mathcal{F}_t\right]. \end{aligned}$$

Since $(i(t))_{t>0}$ and $(r(t))_{t>0}$ follow a normal distribution, $\int_t^T r(u)du$ and $\int_t^T i(u)du$ are normally distributed as well. Hence, the price $p_{I,s}(t,T)$ is finally computed as the expectation of a log-normal distributed random variable.

We now briefly sketch the (not complicated though rather tedious) derivation of the distribution of $(-\int_t^T r(u)du + \int_t^T i(u)du)$ which is derived from the joint multivariate normal distribution of $(\int_t^T r(u)du, \int_t^T i(u)du)$ which is itself completely determined by the expectation, variance and covariance of $\int_t^T r(u)du$ and $\int_t^T i(u)du$.

Expectation and variance of the processes are derived using the identity $r(s) = e^{-\tilde{\kappa}_r(s-t)}r(t) + \tilde{\theta}_r(1 - e^{-\tilde{\kappa}_r(s-t)}) + \tilde{\sigma}e^{-\tilde{\kappa}_r s}\int_t^s e^{\tilde{\kappa}_r u}dW^r(u)$ for $s > t$ and the fact that $\int_s^t f(u)dW^r(u) \sim \mathcal{N}(0, \int_s^t f^2(u)du)$.

Finally, the covariance of the considered processes is calculated as follows:

$$\begin{aligned} &\mathbb{COV}_Q\left[\int_t^T r(s)\,ds, \int_t^T i(s)\,ds\right] \\ &= \mathbb{COV}_Q\left[\tilde{\sigma}_r e^{-\tilde{\kappa}_r s}\int_t^T\left(\int_t^s e^{\tilde{\kappa}_r u}dW^r(u)\right)ds, \sigma_i e^{-\kappa_i s}\int_t^T\left(\int_t^s e^{\kappa_i u}dW^i(u)\,ds\right)\right] \\ &= \tilde{\sigma}_r\sigma_i\left(\int_t^T\int_t^T \mathbb{COV}_Q\left[e^{-\tilde{\kappa}_r s}\int_t^s e^{\tilde{\kappa}_r u}dW^r(u), e^{-\kappa_i z}\int_t^z e^{\kappa_i u}dW^i(u)\right]dzds\right) \\ &= \tilde{\sigma}_r\sigma_i\rho_{ri}\int_t^T\int_t^T\left(\int_t^{s\wedge z} e^{-\tilde{\kappa}_r(s-u)}e^{-\kappa_i(z-u)}\right)dzds \end{aligned}$$

which can then be computed analytically, as well.

B. Sensitivity analyses — further results

This appendix gives more details with respect to the results explained in Section 6. We display key statistics of the observed real returns considering a single premium investment only.

TABLE B1

SENSITIVITY I.

	Nominal Zero-Bond (%)	Zero + Underlying (%)	iCPPI (%)	Option Based Product (%)	Equity Fund (%)	Inflation Linked Zero-Bond (%)	Zero + Underlying (Historic Floor) (%)	iCPPI (Historic Floor) (%)	Zero + Underlying (Market Floor) (%)	iCPPI (Market Floor) (%)	Zero + Underlying (Linker) (%)	iCPPI (Linker) (%)
5%	0.29	−1.75	−2.89	−2.72	−6.10	1.51	−0.25	−1.12	−0.17	−0.89	0.18	0.00
25%	1.08	−0.22	−1.83	−1.46	−1.78	1.51	0.51	−0.35	0.56	−0.27	0.64	0.00
Median	1.62	1.40	0.01	0.76	1.19	1.51	1.38	0.04	1.45	0.02	1.40	0.00
75%	2.16	3.56	3.81	3.75	4.19	1.51	2.86	2.62	2.91	2.59	2.72	1.31
95%	2.96	7.27	8.24	8.02	8.48	1.51	6.08	7.84	5.96	7.83	5.62	7.72
Expected Return	1.72	3.31	3.59	3.55	3.91	1.51	2.74	3.25	2.72	3.25	2.57	3.11

TABLE B2

SENSITIVITY IIA(B).

	Nominal Zero-Bond (%)	Zero + Underlying (%)	iCPPI (%)	Option Based Product (%)	Equity Fund (%)	Inflation Linked Zero-Bond (%)	Zero + Underlying (Historic Floor) (%)	iCPPI (Historic Floor) (%)	Zero + Underlying (Market Floor) (%)	iCPPI (Market Floor) (%)	Zero + Underlying (Linker) (%)	iCPPI (Linker) (%)
5%	0.29	−1.58	−2.75	−2.56	−5.93	1.81	−0.15	−1.11	−0.13	−1.00	0.21	0.00
	(0.29)	(−2.13)	(−3.10)	(−3.01)	(−6.57)	(1.21)	(−0.39)	(−1.12)	(−0.26)	(−0.82)	(0.13)	(0.00)
25%	1.08	−0.13	−1.74	−1.36	−1.71	1.81	0.60	−0.32	0.60	−0.33	0.74	0.00
	(1.08)	(−0.47)	(−2.10)	(−1.78)	(−2.04)	(1.21)	(0.31)	(−0.40)	(0.40)	(−0.26)	(0.51)	(0.00)
Median	1.62	1.44	0.05	0.80	1.23	1.81	1.46	0.05	1.50	0.01	1.58	0.00
	(1.62)	(1.34)	(−0.11)	(0.72)	(1.15)	(1.21)	(1.20)	(0.01)	(0.31)	(0.00)	(1.19)	(0.00)
75%	2.16	3.50	3.75	3.69	4.14	1.81	2.85	2.74	2.91	2.74	2.96	2.26
	(2.16)	(3.76)	(4.00)	(3.96)	(4.40)	(1.21)	(2.91)	(2.60)	(2.93)	(2.58)	(2.51)	(0.17)
95%	2.96	7.10	8.06	7.86	8.32	1.81	5.80	7.71	5.85	7.75	5.89	7.73
	(2.96)	(7.87)	(8.85)	(8.63)	(9.10)	(1.21)	(6.76)	(8.39)	(6.46)	(8.30)	(5.60)	(8.01)
Expected Return	1.72	3.20	3.47	3.43	3.78	1.81	2.65	3.16	2.68	3.20	2.76	3.17
	(1.72)	(3.67)	(4.05)	(3.99)	(4.36)	(1.21)	(3.06)	(3.65)	(2.92)	(3.60)	(2.49)	(3.27)

TABLE B3

SENSITIVITY IIIA(B).

	Nominal Zero-Bond (%)	Zero + Underlying (%)	iCPPI (%)	Option Based Product (%)	Equity Fund (%)	Inflation Linked Zero-Bond (%)	Zero + Underlying (Historic Floor) (%)	iCPPI (Historic Floor) (%)	Zero + Underlying (Market Floor) (%)	iCPPI (Market Floor) (%)	Zero + Underlying (Linker) (%)	iCPPI (Linker) (%)
5%	0.25	−2.44	−3.73	−3.58	−6.18	1.56	−0.30	−1.16	−0.24	−0.96	0.18	0.00
	(0.34)	(−1.12)	(−2.06)	(−1.89)	(−6.11)	(1.65)	(−0.24)	(−1.10)	(−0.20)	(−0.94)	(0.19)	(0.00)
25%	1.04	−0.59	−2.48	−2.00	−1.82	1.56	0.46	−0.38	0.50	−0.32	0.65	0.00
	(1.13)	(−0.19)	(−1.12)	(−0.80)	(−1.78)	(1.65)	(0.51)	(−0.34)	(0.54)	(−0.30)	(0.68)	(0.00)
Median	1.57	1.33	0.67	0.77	1.20	1.56	1.34	0.03	1.41	0.00	1.43	0.00
	(1.67)	(1.51)	(−0.16)	(−0.82)	(1.21)	(1.65)	(1.40)	(0.04)	(1.45)	(0.01)	(1.49)	(0.00)
75%	2.12	3.76	3.98	3.80	4.24	1.56	2.84	2.55	2.89	2.53	2.81	1.39
	(2.21)	(3.36)	(3.43)	(3.76)	(4.20)	(1.65)	(2.90)	(2.65)	(2.96)	(2.67)	(2.86)	(1.73)
95%	2.91	7.71	8.36	8.12	8.58	1.56	6.13	7.93	6.04	7.93	5.78	7.83
	(3.01)	(6.77)	(8.12)	(8.00)	(8.46)	(1.65)	(6.12)	(7.83)	(6.07)	(7.86)	(5.80)	(7.78)
Expected Return	1.67	3.52	3.72	3.58	3.98	1.56	2.74	3.29	2.73	3.30	2.66	3.19
	(1.77)	(3.08)	(3.45)	(3.59)	(3.88)	(1.65)	(2.75)	(3.23)	(2.75)	(3.25)	(2.68)	(3.15)

TABLE B4

SENSITIVITY IVA(B).

	Nominal Zero-Bond (%)	Zero + Underlying (%)	iCPPI (%)	Option Based Product (%)	Equity Fund (%)	Inflation Linked Zero-Bond (%)	Zero + Underlying (Historic Floor) (%)	iCPPI (Historic Floor) (%)	Zero + Underlying (Market Floor) (%)	iCPPI (Market Floor) (%)	Zero + Underlying (Linker) (%)	iCPPI (Linker) (%)
5%	1.25	−1.46	−2.76	−2.61	−5.24	2.59	0.09	−0.90	0.10	−0.79	0.34	0.00
	(−0.66)	(−2.11)	(−3.03)	(−2.87)	(−7.04)	(0.64)	(−0.82)	(−1.38)	(−0.74)	(−1.12)	(0.06)	(0.00)
25%	2.05	0.41	−1.50	−1.01	−0.83	2.59	1.05	−0.15	1.07	−0.15	1.17	0.00
	(0.13)	(−0.81)	(−2.10)	(−1.78)	(−2.76)	(0.64)	(−0.07)	(−0.63)	(−0.01)	(−0.47)	(0.24)	(0.00)
Median	2.59	2.35	1.68	1.78	2.21	2.59	2.33	0.33	2.37	0.23	2.39	0.03
	(0.66)	(0.50)	(−1.15)	(−0.18)	(0.21)	(0.64)	(0.48)	(−0.15)	(0.53)	(−0.10)	(0.57)	(0.00)
75%	3.14	4.80	5.02	4.84	5.29	2.59	4.28	4.52	4.32	4.57	4.28	4.40
	(1.20)	(2.33)	(2.40)	(2.73)	(3.16)	(0.64)	(1.42)	(0.26)	(1.45)	(0.22)	(1.23)	(0.00)
95%	3.95	8.79	9.45	9.21	9.67	2.59	7.98	9.30	7.95	9.34	7.81	9.31
	(1.98)	(5.71)	(7.05)	(6.92)	(7.38)	(0.64)	(3.96)	(6.03)	(3.75)	(5.85)	(3.06)	(5.19)
Expected Return	2.69	4.56	4.76	4.62	5.02	2.59	4.13	4.57	4.13	4.59	4.09	4.54
	(0.75)	(2.05)	(2.42)	(2.56)	(2.84)	(0.64)	(1.32)	(1.83)	(1.27)	(1.79)	(1.13)	(1.55)